Dimensional reductions of BKP and CKP hierarchies

2001 J. Phys. A: Math. Gen. 34 3447

(http://iopscience.iop.org/0305-4470/34/16/313)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL
J. Phys. A: Math. Gen. **34** (2001) 3447–3459 www.iop.org/Journals/ja PII: S0305-4470(01)18083-2

Dimensional reductions of BKP and CKP hierarchies

Ignace Loris

Dienst Theoretische Natuurkunde, Vrije Universiteit Brussel, Pleinlaan 2, B-1050 Brussel, Belgium

Received 12 October 2000, in final form 16 February 2001

Abstract
A discussion of dimensional reductions, which are not classical symmetry reductions, is made for the BKP and CKP hierarchies of integrable evolution equations. A novel direct method for testing Pfaffian solutions to bilinear identities is presented and applied to these reductions.

PACS number: 0230J

AMS classification scheme numbers: 58F07

1. Introduction

Symmetry methods are widely used in the study of ordinary and partial differential equations (PDEs). Solving the question of which transformation (of both dependent and independent variables) leaves the solution space invariant can provide the key to finding the complete solution. The technique may give access to a linearizing transformation or show how to solve certain ordinary differential equations (ODEs) by quadrature only. In the case of PDEs the method is often used in the search for classes of special solutions: symmetry reductions reduce the number of independent variables which feature in an equation. Thus, a PDE may be reduced to an ODE. Solutions of this ODE then yield solutions to the original equation. For example, if a (1 + 1)-dimensional equation for the field u is invariant under both x and t translations (corresponding to symmetries u_x and u_t), one may look for solutions $u(z)$ which depend only upon the combination $z = x - ct$ (travelling wave solutions). In this manner the PDE is reduced to an ODE in the single independent variable z via the condition $u_t + cu_x = 0$.

However, it is known that symmetry reductions cannot account for all the dimensional reductions of PDEs (and hence cannot account for all special solutions). Using a direct method Clarkson and Kruskal [1] succeeded in finding dimensional reductions which are not symmetry reductions in the above sense. A group theoretical explanation was however soon found [2]: one has to look for symmetries of a subset of the solutions (conditional symmetries). The corresponding group action then either takes a solution of the PDE out of the solution space or leaves it invariant (hence the conditional symmetry can still be used for reduction purposes).

The Kadomtsev–Petviashvili (KP) hierarchy of nonlinear PDEs is a (2 + 1)-dimensional integrable hierarchy. It contains many integrable (1 + 1)-dimensional systems such as the Korteweg–de Vries, Boussinesq and nonlinear Schrödinger equations as symmetry reductions.

Some are mere standard symmetry reductions of the type $u_{t_k} = 0$ (where t_k is a specific KP-independent variable); others couple the KP linear problem ('Lax pair') with the nonlinear equations via so-called eigenfunction symmetry reductions. These have been discussed at great length [3–13]. Similar reductions for the BKP and CKP hierarchies have also been described [14, 15].

In this paper I shall investigate some systems which appear to be symmetry reductions at first sight, but are not on closer inspection. Consider, for example, the system

$$q_{i,t} = q_{i,3x} + 3[q_1^2 + q_2^2 + \cdots + q_n^2]q_{i,x} \qquad i = 1, \ldots, n \tag{1}$$

for which Pfaffian-type solutions are discussed in [16]. To this end it is written as

$$\begin{aligned} q_{i,t} &= q_{i,3x} + 3uq_{i,x} \qquad i = 1, \ldots, n \\ u &= q_1^2 + q_2^2 + \cdots + q_n^2. \end{aligned} \tag{2}$$

This way of writing the system and the existence of Pfaffian solutions (see the last section for this terminology) may lead one to think there is a connection with the BKP hierarchy: the BKP linear problem contains the first equation of system (2) and the BKP hierarchy has solutions in Pfaffian form. However it is wrong to think that the system (1) can be obtained from the (2+1)-dimensional BKP hierarchy through a symmetry reduction of type $u_x = (q_1^2 + q_2^2 + \cdots + q_n^2)_x$, since the right-hand side of this expression is not a symmetry of the BKP equation. As will be shown this system can nevertheless be retrieved from the BKP hierarchy via a symmetry method.

In the following sections we shall first introduce the KP hierarchy (with its infinite number of time variables corresponding to an infinite number of symmetries). KP eigenfunctions are introduced and used to describe a class of eigenfunction symmetries. This material serves a basis for the discussion of certain dimensional reductions of the CKP hierarchy in section 3. In section 4 a class of reductions of the BKP hierarchy is discussed. Furthermore a bilinear formulation (bilinear identity containing the Hirota equations) for the reduced systems will be derived.

As a final point we shall introduce a novel way of checking the existence of Pfaffian solutions to the BKP and reduced BKP equations. Existing techniques for the verification of such solutions rely on the transformation of the Hirota differential equations to identities for Pfaffian expressions. The method described in section 5 uses the bilinear identities directly, without the need of writing them out as differential equations. In this manner, all differential equations are verified in one instance. This techniques is illustrated for both the BKP and reduced BKP bilinear hierarchies. This method is reminiscent of that one introduced in [7, 10].

2. KP equations

The KP Lax operator is the first-order pseudo-differential operator $L = \partial + u_2\partial^{-1} + u_3\partial^{-2} + \cdots$ with coefficients $u_2(\underline{t}), u_3(\underline{t}), \ldots$ depending on the independent variables $t_1 = x, t_2, t_3, \ldots$. Multiplication of pseudo-differential operators is defined by $\partial^n \cdot f = \sum_{k \geqslant 0} \binom{n}{k} f_{kx} \partial^{n-k}$ for both positive and negative integer n. The evolution of the fields $u_i(\underline{t})$ is defined through the differential operators $B_n = (L^n)_+$ (the differential part of L^n, n a positive integer, such that $B_1 = \partial$, $B_2 = \partial^2 + 2u_2$, $B_3 = \partial^3 + 3u_2\partial_x + 3(u_3 + u_{2,x}), \ldots$):

$$L_{t_n} = [B_n, L]. \tag{3}$$

It is easily seen that these equations imply the relations

$$\partial_{t_n} B_m - \partial_{t_m} B_n = [B_n, B_m]. \tag{4}$$

It is important to stress that the evolutions (3) are mutually compatible:

$$
\begin{aligned}
(L_{t_m})_{t_n} - (L_{t_n})_{t_m} &= [B_n, B_m]L + B_m[B_n, L] - [B_n, L]B_k - L[B_n, B_m] \\
&\quad - B_n[B_m, L] + [B_m, L]B_n \\
&= [[B_n, B_m], L] + [[L, B_n], B_m] + [[B_m, L], B_n] = 0.
\end{aligned} \tag{5}
$$

The equations (4) at $m = 3$ and $n = 2$ yield the following nonlinear PDE (the KP equation) [17, 18]:

$$4u_{2,x,t_3} - 3u_{2,2t_2} - u_{2,4x} - 12(u_2 u_{2,x})_x = 0. \tag{6}$$

Eigenfunctions Φ (respectively adjoint eigenfunctions Φ^*) are the fields which satisfy the linear evolution equations

$$\Phi_{t_n} = B_n \Phi \qquad (\text{resp.} \quad \Phi^*_{t_n} = -B_n^* \Phi^*). \tag{7}$$

The compatibility condition of these relations is exactly relation (4).

As shown, the flow $L_{t_n} = [B_n, L]$ is compatible with the other flows of the same type. Another compatible evolution (or symmetry) is the following: introduce an eigenfunction Φ and an adjoint eigenfunction Φ^* and define an evolution of L by

$$L_y = [A, L] \tag{8}$$

with $A = \Phi\partial^{-1}\Phi^* \equiv \Phi\Phi^*\partial^{-1} - \Phi\Phi^*_x\partial^{-2} + \Phi\Phi^*_{2x}\partial^{-3} + \cdots$. The compatibility of this evolution with the evolution (3) follows from $(\Phi\partial^{-1}\Phi^*)_{t_n} = ([B_n, \Phi\partial^{-1}\Phi^*])_-$ [11, lemma 2.1] and from $B_{n,y} \equiv ((L^n)_+)_y = ((L^n)_y)_+ = ([\Phi\partial^{-1}\Phi^*, L^n])_+ = [\Phi\partial^{-1}\Phi^*, B_n])_+$:

$$(L_y)_{t_n} - (L_{t_n})_y = [A, L]_{t_n} - [B_n, L]_y = [[B_n, A], L] + [[L, B_n], A] + [[A, L], B_n] = 0. \tag{9}$$

The reader can check that $u_2 + \epsilon u_{2,y} = u_2 - \epsilon(\Phi\Phi^*)_x$ satisfies the KP equation (6) up to first order in ϵ, showing explicitly that $(\Phi\Phi^*)_x$ is a symmetry of the KP equation (6). These eigenfunction symmetries are well known [3, 19] and go back to the squared eigenfunctions of the KdV equation [20].

It should be clear to the reader that one can also introduce compatible evolutions (i.e. symmetries) for the KP hierarchy by taking linear combinations of the previous cases (e.g. $A = \Phi_1\partial^{-1}\Phi_1^* + \Phi_2\partial^{-1}\Phi_2^*$, etc), i.e. symmetries have a linear structure.

A standard method of reducing PDEs is imposing a condition of type $u_t = 0$ where u_t is a symmetry of the equation. The $(2 + 1)$-dimensional KP hierarchy may be reduced in this way to a $(1 + 1)$-dimensional one. For example, $L_{t_2} = 0$ ($\Leftrightarrow$ $[B_2, L] = 0$ or $B_2 = L^2$) or $L_x + L_y = 0$ (that is $[B_1 + \Phi\partial^{-1}\Phi^*, L] = 0$ or $L = B_1 + \Phi\partial^{-1}\Phi^*$). These types of reduction result in the KdV hierarchy, the nonlinear Schrödinger equation, etc and have been investigated in detail [3, 5–12, 14, 15].

In the two following sections, we shall discuss BKP and CKP hierarchies and some of their dimensional reductions. In both cases a condition will be put on the KP Lax operator and its adjoint. This will imply a relation between eigenfunctions and adjoint eigenfunctions. It follows that the eigenfunction symmetries (depending on eigenfunctions and adjoint eigenfunctions) will take a different form in both cases.

3. The CKP case

The CKP hierarchy is obtained from the KP hierarchy by imposing the following condition on the KP Lax operator [21, 22]:

$$L^* + L = 0 \tag{10}$$

and by considering evolution with respect to odd-indexed time variables $t_1, t_3, t_5, \ldots$ only. The condition (10) implies $u_3 = -\frac{1}{2}u_{2,x}$, $u_5 = -\frac{3}{2}u_{4,x} + \frac{1}{4}u_{2,4x}$, etc. At the same time it follows that the differential operators B_n are anti-symmetric if n is odd, and symmetric if n is even. This then implies that there is no distinction between the CKP eigenfunction and the adjoint eigenfunction (i.e. they satisfy the *same* linear equations). For further reference the first member of the linear CKP problem is

$$\Phi_{t_3} = B_3\Phi = \Phi_{3x} + 3u_2\Phi_x + \tfrac{3}{2}u_{2,x}\Phi. \tag{11}$$

It is easy to see that the condition (10) remains valid under the evolutions $L_{t_n} = [B_n, L]$ with n odd: $L^*_{t_n} + L_{t_n} = [B_n, L]^* + [B_n, L] = -[B_n, L] + [B_n, L] = 0$; it is not preserved under evolutions $L_{t_n} = [B_n, L]$ with n even ($L^*_{t_n} + L_{t_n} = [B_n, L]^* + [B_n, L] = 2[B_n, L] \neq 0$, in general). This is the reason why the CKP hierarchy only contains odd-numbered time variables.

Just as $\partial_{t_2}, \partial_{t_4}, \ldots$ do not preserve the condition (10), so is the case with the evolutions of type $L_y = [A, L]$ where $A = q\partial^{-1}r$ (with both q and r CKP eigenfunctions). Indeed $0 = L^*_y + L_y = [A, L]^* + [A, L] = [A + A^*, L]$. Only through the special choice $q = r$ can this be guaranteed ($A^* = -q\partial^{-1}q = -A$). Other possibilities include the combination $A = q\partial^{-1}q + r\partial^{-1}r$ or $A = q\partial^{-1}r + r\partial^{-1}q$, etc.

These eigenfunction symmetries may be used in symmetry reductions of the CKP hierarchy, coupling the linear equations for q and r with the evolution equations for the fields $u_2, \ldots$. For example, setting $L_x + L_y = 0$ (i.e. $L = B_1 + q\partial^{-1}q$ such that $[B_1 + A, L] = 0$) leads to the relation $u_2 = q^2$ and hence (via equation (11)) to the modified KdV equation. This procedure is described in [15].

As already mentioned the flow ∂_{t_2} is not admissible in the CKP setting as $L^*_{t_2} + L_{t_2} = 2[B_2, L] \neq 0$ except in the case where one would impose the reduction condition $B_2 = L^2$ (such that $[B_2, L] = 0$ and $L_{t_2} = 0$). This condition implies that $2u_4 + u_{3,x} + u_2^2 = 0$ and hence that $u_{2,t_3} = \frac{1}{4}u_{2,3x} + 3u_2u_{2,x}$ (the KdV equation). Similarly one may put $B_4 = L^4$ and find that the (2 + 1)-dimensional CKP equation reduces to a third-order (1 + 1)-dimensional system for the fields u_2 and u_4.

In the case of the eigenfunction-related evolutions, one may consider a symmetric combination $A = q\partial^{-1}r - r\partial^{-1}q$ instead of the anti-symmetric $q\partial^{-1}r + r\partial^{-1}q$. It is not associated with a symmetry of the CKP hierarchy, but can nevertheless be used in reductions: let us, for example, set $L_y = [q\partial^{-1}r - r\partial^{-1}q, L]$ such that $L^*_y + L_y = 2[q\partial^{-1}r - r\partial^{-1}q, L]$, at the same time setting $L_y = 0$ by imposing the constraint $L^{-2} = q\partial^{-1}r - r\partial^{-1}q$ (both sides are symmetric operators of order -2). This condition yields $1 = -qr_x + rq_x$ and $-2u_2 = rq_{3x} - qr_{3x}$. It then follows that $q_{t_3} = q_{3x} + 3u_2q_x + \frac{3}{2}u_{2,x}q = \frac{1}{4}q_{3x} - \frac{3}{4}q_xq_{2x}/q$; this equation is transformed into the modified KdV equation by the transformation $q = \exp v$.

Another possibility would be to consider the evolution $L_y = [B_2 + A, L]$ (still with $A = q\partial^{-1}r - r\partial^{-1}q$) such that $L^*_y + L_y = 2[B_2 + A, L]$ ($\neq 0$), indicating again incompatibility with the condition (10). However, imposing the constraint $L^2 = B_2 + q\partial^{-1}r - r\partial^{-1}q$ (implying $[B_2 + A, L] = 0$ and $L_y = 0$) solves this problem by reducing the CKP hierarchy. This particular condition implies $\partial^2 + 2u_2 + (2u_4 - \frac{1}{2}u_{2,2x} + u_2^2)\partial^{-2} + \cdots = \partial^2 + 2u_2 + (-qr_x + q_xr)\partial^{-2} + \cdots$ or $(2u_4 - \frac{1}{2}u_{2,2x} + u_2^2) = (-qr_x + q_xr)$; together with $L_{t_3} = [B_3, L]$ (containing $u_{2,t_3} = 6u_2u_{2,x} + 3u_{4,x} - \frac{1}{2}u_{2,3x}$), this yields the following (1 + 1)-dimensional system for the fields u_2, q, r:

$$\begin{aligned} u_t &= \tfrac{1}{4}u_{3x} + 3uu_x - \tfrac{3}{2}(qr_x - q_xr)_x \\ q_t &= q_{3x} + 3uq_x + \tfrac{3}{2}u_xq \\ r_t &= r_{3x} + 3ur_x + \tfrac{3}{2}u_xr. \end{aligned} \tag{12}$$

This is a coupled KdV system. Solutions of this system are reported in [23]. The case $q = r = 0$ (i.e. CKP with $L^2 = B_2$) yields the KdV equation.

The condition $L^4 = B_4 + q\partial^{-1}r - r\partial^{-1}q$ leads in a similar way to a coupled system:

$$\begin{aligned}
u_{2,t_3} &= -\tfrac{1}{2}u_{2,3x} + 6u_2u_{2,x} + 3u_{4,x} \\
u_{4,t_3} &= \tfrac{1}{4}u_{4,3x} - 6u_2u_{4,x} + 3u_{2,x}u_{2,2x} + \tfrac{9}{4}u_2u_{2,3x} - 3(u_2^3)_x + \tfrac{3}{4}(rq_x - qr_x)_x \\
q_{t_3} &= q_{3x} + 3uq_x + \tfrac{3}{2}u_xq \\
r_{t_3} &= r_{3x} + 3ur_x + \tfrac{3}{2}u_xr.
\end{aligned} \tag{13}$$

In conclusion, the constraint

$$L^k = B_k + q\partial^{-1}r - r\partial^{-1}q \qquad k\text{: even integer} \tag{14}$$

(q, r CKP eigenfunctions) on the CKP Lax operator L yields a number of $(1+1)$-dimensional nonlinear evolution equations. I wish to emphasize that the above dimensional reductions are not symmetry reductions in the classical sense, since the evolutions introduced are not compatible with the CKP condition (10). The compatibility of the equations introduced for the extra evolution parameter (y) can only be guaranteed if one at the same imposes that the evolution is trivial ($L_y = 0$). The result nevertheless is a series of integrable reductions.

4. The BKP case

Analogous to formula (10), the BKP hierarchy is obtained by imposing the condition

$$L^*\partial + \partial L = 0 \tag{15}$$

on the KP Lax operator. Again, one must at the same time forget about the even-indexed KP time variables as they do not preserve this condition. Formula (15) implies $u_3 = -u_{2,x}$, etc such that $B_n^*\partial + \partial B_n = 0$ when n is odd. This implies that a BKP eigenfunction Φ gives rise to a BKP adjoint eigenfunction Φ_x and that $\Phi =$ constant is always a BKP eigenfunction. The basic member of the BKP linear system is

$$\Phi_{t_3} = \Phi_{3x} + 3u_2\Phi_x. \tag{16}$$

For n even one has $B_n^*\partial = \partial B_n + 2a_n$ (where $a_n = \mathrm{Res}_\partial(L^n)$ and Res_∂ is the coefficient of ∂^{-1}). The most basic equation contained in the BKP hierarchy is

$$9v_{x,t_5} - 5v_{2t_3} + (-5v_{2x,t_3} - 15v_xv_{t_3} + v_{5x} + 15v_xv_{3x} + 15v_x^3)_x = 0 \tag{17}$$

with $v_x = u_2$.

The following section deals with the definition of some new dimensional reductions of the BKP hierarchy. In the second section, these reductions will be cast in a bilinear form. Such a bilinear formulation is very convenient for the derivation of classes of exact solutions to the nonlinear PDEs.

4.1. Nonstandard reductions

The standard reductions $L_{t_k} = 0$ ($L^k = B_k$) with $k = 3, 5, \ldots$ include the Sawada–Kotera equation [22, 24]. Consider two BKP eigenfunctions q and r and hence two adjoint BKP eigenfunctions q_x and r_x. A tau-function description of the eigenfunction symmetry reductions $L^k = B_k + q\partial^{-1}r_x - r\partial^{-1}q_x$ ($k = 1, 3, \ldots$) is given in [14] ($L_y = [B_k + q\partial^{-1}r_x - r\partial^{-1}q_x, L]$ is an evolution compatible with BKP).

Now, consider the following evolution with respect to a parameter y: $L_y = [A, L]$ with $A = B_k + q\partial^{-1}r_x + r\partial^{-1}q_x$ and where k is an even integer and q and r are two BKP eigenfunctions. From the condition (15) it should follow that $0 = L_y^*\partial + \partial L_y =$

$[L^*, A^*]\partial - \partial[A, L]$ or that $[A + \partial^{-1}A^*\partial, L] = 0$. This corresponds to $2[B_k + q\partial^{-1}r_x + r\partial^{-1}q_x - \partial^{-1}(qr)_x + \partial^{-1}a_k, L] = 0$. This condition is met if one imposes the following constraint:

$$L^k = B_k + q\partial^{-1}r_x + r\partial^{-1}q_x. \tag{18}$$

The above constraint (18) on the BKP Lax operator reduces the (2 + 1)-dimensional BKP hierarchy to (1 + 1)-dimensional integrable hierarchies. In this expression q and r are BKP eigenfunctions (i.e. they satisfy $\Phi_{t_n} = B_n\Phi$ as, for example, in (16)).

Taking for example $k = 2$ in (18) with $L^2 = \partial^2 + 2u_2 - u_{2,x}\partial^{-1} + \cdots$, one finds $u_{2,x} = -(qr)_x$. If zero boundary conditions are assumed this implies $u_2 = -qr$ and hence (via (16)):

$$q_{t_3} = q_{3x} - 3qrq_x \qquad r_{t_3} = r_{3x} - 3qrr_x. \tag{19}$$

The special case $q = r$ (or $L^k = B_k + 2q\partial^{-1}q_x$) yields the modified KdV equation $q_{t_3} = q_{3x} - 3q^2q_x$ and choosing $q = r = 0$ (corresponding to the two-reduction $B_2 = L^2$) implies $u_{2,x} = 0$ and hence does not contain any interesting system. The choice $r = 1$ (remember that $r =$ constant is a valid BKP eigenfunction) yields the KdV equation $q_t = q_{3x} - 3qq_x$.

In a completely similar fashion one may introduce the reduction condition $L^2 = B_2 - 2\sum_{i=1}^n q_i\partial^{-1}q_{i,x}$ (q_i BKP eigenfunctions) which implies $u_2 = \sum_{i=1}^n q_i^2$ (up to a constant) and hence

$$q_{i,t} = q_{i,3x} + 3\left(\sum_{i=1}^n q_j^2\right)q_{i,x} \qquad i = 1, \ldots, n. \tag{20}$$

The manner in which the system (1) is contained in the BKP hierarchy is hereby explained; it is a constraint of the BKP hierarchy corresponding to the conditional symmetry $L_y = [B_2 - 2\sum_{i=1}^n q_i\partial^{-1}q_{i,x}, L]$.

Nonzero boundary conditions $u_2 = \sum_{i=1}^n q_i^2 + c$ are actually a special case of the vector constraint $u_2 = \sum_{i=1}^{n+1} q_i^2$ (as $q_{n+1} = \sqrt{c}$ is a valid BKP eigenfunction).

The reduction condition $L^4 = B_4 + q\partial^{-1}q_x$ leads to the relation

$$-6u_2u_{2,x} - 2u_{4,x} + u_{2,3x} = qq_x. \tag{21}$$

The equation $L_{t_3} = [B_3, L]$ implies $u_{2,t_3} = 6u_2u_{2,x} + 3u_{4,x} - 2u_{2,3x}$. This in turn implies the following coupled KdV system [25, 26]:

$$\begin{aligned} u_{2,t_3} &= -\tfrac{1}{2}u_{2,3x} - 3u_2u_{2,x} - \tfrac{3}{4}(q^2)_x \\ q_{t_3} &= q_{3x} + 3u_2q_x. \end{aligned} \tag{22}$$

The case $q = 0$ (i.e. $L^4 = B_4$) reduces to the Korteweg–de Vries equation [27].

In the following sections we shall transform the condition (18) into a bilinear constraint on the BKP tau-function. First we introduce this tau-function and the accompanying BKP bilinear identity.

4.2. Bilinear constraints

The nonlinear equations for the BKP hierarchy can be summarized into a bilinear identity [18, 22]. One shows that if $\psi(\underline{t}, \lambda)$ is a BKP eigenfunction which also satisfies the eigenvalue problem $L\psi = \lambda\psi$ one has

$$\mathrm{Res}_\lambda\left[\lambda^{-1}\psi(\underline{t}, \lambda)\psi(\underline{t}', -\lambda)\right] = 1 \tag{23}$$

(Res_λ = coefficient of λ^{-1}). At the same time it is shown that the wavefunction ψ can be represented as $\psi(\underline{t},\lambda) = \tau(\underline{t}-\underline{\epsilon}(\lambda))/\tau(\underline{t})\exp\xi(\underline{t},\lambda)$ where $\xi(\underline{t},\lambda) = \sum_{n=1,3,\ldots}\lambda^n t_n$ and $\underline{\epsilon}(\lambda) = (2\lambda^{-1}, 2\lambda^{-3}/3, 2\lambda^{-5}/5, \ldots)$. This then implies the equation

$$\mathrm{Res}_\lambda\big[\lambda^{-1}\tau(\underline{t}-\underline{\epsilon}(\lambda))\tau(\underline{t}'+\underline{\epsilon}(\lambda))\exp\xi(\underline{t}-\underline{t}',\lambda)\big] = \tau(\underline{t})\tau(\underline{t}') \tag{24}$$

for the BKP tau-function τ. This equation contains the Hirota bilinear equations for all the nonlinear PDEs in the BKP hierarchy. The lowest-order equation (corresponding to equation (17)) in this set is

$$(9D_1D_5 - 5D_1^3D_3 - 5D_3^2 + D_1^6)\tau\cdot\tau = 0 \tag{25}$$

where D_i the Hirota D-operator with respect to t_i [28]. All the fields u_i can be expressed in terms of τ. The field u_2 is connected to the tau-function by the relation $u_2 = 2\partial_x^2\log\tau$.

A condition of the type (18) can also be transformed into a bilinear identity for the function τ and the fields $\rho_1 = q\tau$ and $\rho_2 = r\tau$. A similar derivation may be found in [14] for the k-constrained BKP system (with k odd).

Proposition. *The k-constrained BKP hierarchy (k : 2, 4, ...), defined by the condition*

$$L^k = B_k + 2q\partial^{-1}r_x + 2r\partial^{-1}q_x \tag{26}$$

is equivalent to the bilinear identity

$$\mathrm{Res}_\lambda\big[\lambda^{k-1}\tau(\underline{t}-\underline{\epsilon}(\lambda))\tau(\underline{t}'+\underline{\epsilon}(\lambda)\mathrm{e}^{\xi(\underline{t}-\underline{t}',\lambda)}\big] = -2[\rho_1(\underline{t})\rho_2(\underline{t}') + \rho_1(\underline{t}')\rho_2(\underline{t})] \tag{27}$$

for the BKP tau-function τ and the fields $\rho = q\tau$ and $\sigma = r\tau$.

Proof. Relation (26) corresponds to

$$L^k = B_k + 2\sum_{m=1}^{\infty}(-)^{m+1}(qr_{mx} + rq_{mx})\partial^{-m}. \tag{28}$$

It follows that

$$\mathrm{Res}_\partial[L^k\partial^m] = 2(-)^m(qr_{(m+1)x} + rq_{(m+1)x}) \qquad m \geqslant 0 \tag{29}$$

or, on using $L = P\partial P^{-1}$ with $P = 1 + w_1\partial^{-1} + w_2\partial^{-2} + \cdots$ [18],

$$\mathrm{Res}_\partial[P\partial^k P^{-1}\partial^m] = 2(-)^m(qr_{(m+1)x} + rq_{(m+1)x}). \tag{30}$$

The BKP condition (15) corresponds to $P^* = \partial P^{-1}\partial^{-1}$ and hence one has that

$$\mathrm{Res}_\partial[P\partial^{k-1}P^*\partial^{m+1}] = (-)^m(qr_{(m+1)x} + rq_{(m+1)x}). \tag{31}$$

Using the lemma [29, p 82] on the left-hand side, one finds

$$\mathrm{Res}_\lambda[\lambda^{k-1}P(\lambda)(-)^{m+1}(\partial^{m+1}P)(-\lambda)] = 2(-)^m(qr_{(m+1)x} + rq_{(m+1)x}) \tag{32}$$

with $P(\lambda) = 1 + w_1/\lambda + w_2/\lambda^2 + \cdots$. As $\psi(\underline{t},\lambda) = P\exp\xi(\underline{t},\lambda)$, one has [18]

$$\mathrm{Res}_\lambda[\lambda^{k-1}\psi(\underline{t},\lambda)\partial_x^{m+1}\psi(\underline{t},-\lambda)] = -2q(\underline{t})r_{(m+1)x}(\underline{t}) - 2r(\underline{t})q_{(m+1)x}(\underline{t}) \tag{33}$$

(for $m \geqslant 0$). The case $m = 0$ implies $\partial_x\mathrm{Res}_\lambda[\lambda^{k-1}\psi(\underline{t},\lambda)\psi(\underline{t},-\lambda)] = -4(qr)_x$. In the case of zero boundary conditions this leads to relation (33) for $m \geqslant -1$. This implies that

$$\mathrm{Res}_\lambda[\lambda^{k-1}\psi(\underline{t},\lambda)\psi(\underline{t}',-\lambda)] = -2q(\underline{t})r(\underline{t}') - 2q(\underline{t}')r(\underline{t}). \tag{34}$$

Setting $q = \rho_1/\tau$ and $r = \rho_2/\tau$ one finds relation (27). □

The cases k odd (see [14]) and k even of the constrained BKP hierarchy

$$L^k = B_k + 2q\partial^{-1}r_x + 2(-)^k r\partial^{-1}q_x \tag{35}$$

can be combined into the single bilinear identity

$$\mathrm{Res}_\lambda\big[\lambda^{k-1}\tau(\underline{t}-\underline{\epsilon}(\lambda))\tau(\underline{t}'+\underline{\epsilon}(\lambda))\mathrm{e}^{\xi(\underline{t}-\underline{t}',\lambda)}\big] = -2[\rho_1(\underline{t})\rho_2(\underline{t}') + (-)^k\rho_1(\underline{t}')\rho_2(\underline{t})]. \tag{36}$$

Expansion of this expression in powers of $\underline{t}-\underline{t}'$ yields a set of Hirota equations describing the coupling between the bilinear fields τ, ρ_1 and ρ_2. Solutions to the equation (36) are discussed in the following section.

5. Solutions of (reduced) BKP bilinear equations

In this section a new method for showing Pfaffian solutions of the BKP and reduced BKP hierarchies (24) and (36) is introduced. To prove such solutions, one traditionally starts from a suitable KP tau-function and then uses its connection with BKP tau-functions: a BKP tau-function is the square root of a KP tau-function (after setting $t_{2n} = 0$) [18]. One can also directly check this type of solution on the lowest-order bilinear equations ((25), ...) contained in the hierarchy [30]. In the following paragraphs I wish to introduce a method which allows for the direct verification of the Pfaffian expressions on the BKP bilinear identity (24). Besides the obvious advantage of not having to refer to other equations and to the nontrivial link between KP and BKP tau-function, there exists the possibility of using this same technique on the reduced BKP bilinear identities (36). Furthermore all PDEs in the hierarchy are verified at once, using only elementary properties of Pfaffians.

Consider two BKP eigenfunctions f and g and define the BKP eigenfunction potential [14] by

$$\Omega(\underline{t} \pm \underline{\epsilon}(\lambda)) = \Omega(\underline{t}) + f(\underline{t} \pm \underline{\epsilon}(\lambda))g(\underline{t}) - f(\underline{t})g(\underline{t} \pm \underline{\epsilon}(\lambda)) \tag{37}$$

meaning

$$\begin{aligned}
&\Omega_x(f, g) = f_x g - f g_x \\
&\Omega_{t_3}(f, g) = f_{3x} g - f g_{3x} - 2 f_{2x} g_x + 2 f_x g_{2x} + 3u_2(f_x g - f g_x) \\
&\dots .
\end{aligned} \tag{38}$$

A second concept needed is the Pfaffian. In general a Pfaffian $(1, 2, \dots, N)$ of size N (N an even integer) can (recursively) be defined by the expansion rule

$$(1, 2, \dots, N) = \sum_{i=2}^{N} (-)^i (1, i)(2, 3, \dots, \not{i}, \dots, N) \tag{39}$$

where (i, j) (with $i < j : 1, \dots, N$) are $N(N-1)/2$ independent quantities (see also [27, 30, 31]). One has, for example, that $(1, 2, 3, 4) = (1, 2)(3, 4) - (1, 3)(2, 4) + (1, 4)(2, 3)$, etc. The square of the Pfaffian (39) equals the determinant of the anti-symmetric matrix with above-diagonal elements (i, j). The Pfaffian with repeated indices (e.g. $(1, 1, 2, \dots, N-1)$) is zero.

Below, we shall show that if one defines the quantity (i, j) by the eigenfunction potential $\Omega(f_i, f_j)$ the Pfaffian turns into a BKP tau-function:

$$\tau = (1, 2, \dots, N) \qquad \text{with} \quad (i, j) = \Omega(f_i, f_j) \tag{40}$$

satisfies the bilinear identity (24). Here the f_i ($i : 1 \dots N$) are vacuum BKP eigenfunctions: $f_{i,t_n} = f_{i,nx}$. The potentials $\Omega(f_i, f_j)$ all carry an arbitrary integration constant.

First of all, one needs expressions for $\tau(\underline{t} \pm \underline{\epsilon}(\lambda))$. To this end we introduce an auxiliary vacuum BKP eigenfunction $f_0 = 1$ such that $\Omega(f, f_0) = \int f_x = f$ (or $(i, 0) = f_i(\underline{t})$) and the symbols $(S, i) = f_i(\underline{t} - \underline{\epsilon}(\lambda))$ and $(\bar{S}, i) = f_i(\underline{t} + \underline{\epsilon}(\lambda))$ and $(S, 0) = (\bar{S}, 0) = 1$.

Lemma 1. *One has for* $\tau(\underline{t}) = (1, 2, \dots, N)$

$$\tau(\underline{t} - \underline{\epsilon}(\lambda)) = (S, 0, 1, \dots, N) \qquad \tau(\underline{t} + \underline{\epsilon}(\lambda)) = (\bar{S}, 0, 1, \dots, N). \tag{41}$$

The proof of this statement goes by induction on N. The case $N = 2$ reduces to $\Omega_{12}(\underline{t} - \underline{\epsilon}(\lambda)) = \Omega_{12}(\underline{t}) + f_1(\underline{t}) f_2(\underline{t} - \underline{\epsilon}(\lambda)) - f_1(\underline{t} - \underline{\epsilon}(\lambda)) f_2(\underline{t})$ which is formula (37). For general N, one may first expand τ as

$$\tau = (1, 2, \dots, N) = \sum_{j=2}^{N} (-)^j \Omega_{1j} (2, \dots, \not{j}, \dots, N) \tag{42}$$

and hence by formula (37) and the inductive step

$$
\begin{aligned}
\tau(\underline{t}-\underline{\epsilon}(\lambda)) &= \sum_{j=2}^{N}(-)^j[\Omega_{1j}+(S,1)(0,j)-(0,1)(S,j)](S,0,2,\ldots,\not{j},\ldots,N) \\
&= (S,0,1,\ldots,N)-(1,S)(0,2,\ldots,N)+(1,0)(S,2,\ldots,N) \\
&\quad +(S,1)[-(0,S,0,2,\ldots,N)-(S,0)(0,2,\ldots,N)-(0,0)(S,2,\ldots,N)] \\
&\quad -(1,0)[(S,S,0,2,\ldots,N)+(S,0)(S,2,\ldots,N)-(S,S)(0,2,\ldots,N)] \\
&= (S,0,1,\ldots,N)
\end{aligned} \tag{43}
$$

where we have again used the expansion theorem of Pfaffians and the conventions $(S,0)=1$, $(0,0)=(S,S)=0$.

We shall use the expression (41) in expanded form:

$$
\begin{aligned}
\tau(\underline{t}-\underline{\epsilon}(\lambda)) &= \sum_{i=0}^{N}(-)^i(S,i)(0,1,\ldots,\not{i},\ldots,N) \\
\tau(\underline{t}'+\underline{\epsilon}(\lambda)) &= \sum_{j=0}^{N}(-)^i(\bar{S},j)(0,1,\ldots,\not{j},\ldots,N)'.
\end{aligned} \tag{44}
$$

Another formula needed in the derivation of Pfaffian solutions of the BKP bilinear identity is the following.

Lemma 2.

$$\mathrm{Res}_\lambda\big[\lambda^{-1}f(\underline{t}-\underline{\epsilon}(\lambda))g(\underline{t}'+\underline{\epsilon}(\lambda))\exp\xi(\underline{t}-\underline{t}',\lambda)\big]=f(\underline{t})g(\underline{t}')-2\Omega(f,g)+2\Omega(f,g)'. \tag{45}$$

This lemma is shown in [14, appendix] for general BKP eigenfunctions f and g. Here we only need it for vacuum BKP eigenfunctions f and g ($f_{t_n}=f_{nx}$ and $g_{t_n}=g_{nx}$). Note that both left- and right-hand sides are free of integration constants.

Now it becomes quite easy to check that the Pfaffian-type expression (40), with definition (37) and (38), is a solution to the bilinear identity (24).

Theorem 1. *Let $\tau=(1,2,\ldots,N)$, with $(i,j)=\Omega(f_i,f_j)$ and $f_{i,t_n}=f_{i,nx}$, then one has*

$$\mathrm{Res}_\lambda\big[\lambda^{-1}\tau(\underline{t}-\underline{\epsilon}(\lambda))\tau(\underline{t}'+\underline{\epsilon}(\lambda)\exp\xi(\underline{t}-\underline{t}',\lambda)\big]=\tau(\underline{t})\tau(\underline{t}'). \tag{46}$$

Proof. By the formulas (44) we have that

$$
\begin{aligned}
&\mathrm{Res}_\lambda\big[\lambda^{-1}\tau(\underline{t}-\underline{\epsilon}(\lambda))\tau(\underline{t}'+\underline{\epsilon}(\lambda)\exp\xi(\underline{t}-\underline{t}',\lambda)\big] \\
&\qquad = \sum_{i,j\geqslant 0}(-)^{i+j}(0,\ldots,\not{i},\ldots,N)(0,\ldots\not{j},\ldots,N)' \\
&\qquad\quad \times\mathrm{Res}_\lambda\big[\lambda^{-1}f_i(\underline{t}-\underline{\epsilon}(\lambda))f_j(\underline{t}'+\underline{\epsilon}(\lambda))\exp\xi(\underline{t}-\underline{t}',\lambda)\big]
\end{aligned} \tag{47}
$$

which is equal by formula (45) to

$$\sum_{i,j\geqslant 0}(-)^{i+j}(0,\ldots,\not{i},\ldots,N)(0,\ldots\not{j},\ldots,N)'[f_i(\underline{t})f_j(\underline{t}')-2(i,j)+2(i,j)']. \tag{48}$$

As $f_0=1$, the first term in square brackets will yield $(1,\ldots,N)(1,\ldots,N)'$ (i.e. $\tau(\underline{t})\tau(\underline{t}')$). The last two terms in square brackets yield nothing as they reduce to

$$2\sum_{j\geqslant 0}(-)^j(j,0,\ldots,N)(0,\ldots\not{j},\ldots,N)'-2\sum_{i\geqslant 0}(-)^i(0,\ldots,\not{i},\ldots,N)(i,0,\ldots,N)' \tag{49}$$

in which $(j,0,\ldots,N)$ and $(i,0,\ldots,N)'$ are zero (repeated indices). □

It is important to stress the presence of arbitrary constants in the eigenfunction potentials $(i, j) = \Omega(f_i, f_j)$ and hence in the BKP tau functions $\tau = (1, 2, \ldots, N)$. For the reduced tau-functions these constants will no longer be (completely) free.

A formula similar to (45) will make the verification of the reduced bilinear identity (36) just as easy.

Lemma 3.

$$\mathrm{Res}_\lambda\big[\lambda^{k-1} f(\underline{t} - \underline{\epsilon}(\lambda))g(\underline{t}' + \underline{\epsilon}(\lambda) \exp \xi(\underline{t} - \underline{t}', \lambda)\big] = -2\Omega(f, g_{kx}) + 2(-)^k \Omega(f_{kx}, g)'. \quad (50)$$

The case of k odd may be proved by taking the t_k-derivative of relation (45). The case of k even may be obtained by taking the x-derivative of the case $k - 1$. Here one should actually write $-2\Omega(f, g_{kx}) + 2(-)^k\Omega(f_{kx}, g)' + 2C_{f,g_{kx}} - 2(-)^k C_{f_{kx},g}$ in the right-hand side of (50) as no integration constants are present in this equality.

Theorem 2. *The reduced BKP bilinear identity (36) has solutions $\tau = (1, \ldots, N)$, $\rho_2 = (0, 1, \ldots, N + 1)$ and $\rho_1 = (0, 1, \ldots .N - 1)$ (with $(i, j) = \Omega(f_i, f_j)$ and $f_{i,t_n} = f_{i,nx}$ and $f_0 = 1$) with the additional constraint $f_{i,kx} = f_{i+1}$ (only for $i > 0$) and the condition $C_{j,l+1} = (-)^k C_{j+1,l}$ between the eigenfunction potentials (see note after proof).*

Proof. One uses the expressions (44) for the shifted tau-function such that

$$\begin{aligned}
&\mathrm{Res}_\lambda\big[\lambda^{k-1}\tau(\underline{t} - \underline{\epsilon}(\lambda))\tau(\underline{t}' + \underline{\epsilon}(\lambda)) \exp \xi(\underline{t} - \underline{t}', \lambda)\big] \\
&\qquad = \sum_{i,j\geqslant 0} (-)^{i+j}(0, \ldots, \not{i}, \ldots, N)(0, \ldots \not{j}, \ldots, N)' \\
&\qquad\quad \times \mathrm{Res}_\lambda\big[\lambda^{k-1} f_i(\underline{t} - \underline{\epsilon}(\lambda)) f_j(\underline{t}' + \underline{\epsilon}(\lambda)\mathrm{e}^{\xi-\xi'}\big] \qquad (51)
\end{aligned}$$

$\mathrm{Res}_\lambda\big[\lambda^{k-1} f_i(\underline{t}-\underline{\epsilon}(\lambda)) f_j(\underline{t}'+\underline{\epsilon}(\lambda))\mathrm{e}^{\xi-\xi'}\big]$ is zero if $i = j = 0$ ($f_0 = 1$) and equal to $-2(i, j+1)$ if $i = 0$ and $2(-)^k(i+1, j)'$ if $j = 0$; otherwise it is $-2(i, j+1)+2(-)^k(i+1, j)'+2C_{i,j+1} - 2(-)^k C_{i+1,j}$ (remember that $f_{i+1} = f_{i,kx}$). If one sets $-C_{i,j+1} + (-)^k C_{i+1,j} = 0$ then the right-hand side of (51) reduces to

$$\begin{aligned}
&2 \sum_{i\geqslant 0, j>0} (-)^{i+j}(0, \ldots, \not{i}, \ldots, N)(0, \ldots \not{j}, \ldots, N)'(-)(i, j + 1) \\
&\qquad +2(-)^k \sum_{i>0, j\geqslant 0} (-)^{i+j}(0, \ldots, \not{i}, \ldots, N)(0, \ldots \not{j}, \ldots, N)'(i + 1, j)' \qquad (52)
\end{aligned}$$

or

$$\begin{aligned}
&2 \sum_{j>0} (-)^j(j + 1, 0, \ldots, N)(0, \ldots \not{j}, \ldots, N)' \\
&\qquad +2(-)^k \sum_{i>0} (-)^i(0, \ldots, \not{i}, \ldots, N)(0, \ldots, N)'. \qquad (53)
\end{aligned}$$

Only one term in each sum remains (no repeated indices in Pfaffians allowed):

$$2(N + 1, 0, \ldots, N)(0, \ldots, N - 1)' + 2(-)^k(0, \ldots, N - 1)(N + 1, 0, \ldots, N)' \qquad (54)$$

or

$$-2(0, \ldots, N, N + 1)(0, \ldots, N - 1)' + 2(-)^{k+1}(0, \ldots, N - 1)(0, \ldots, N, N + 1)' \qquad (55)$$

which is just

$$-2[\rho_1(\underline{t})\rho_2(\underline{t}') + (-)^k \rho_1(\underline{t}')\rho_2(\underline{t})] \qquad (56)$$

thereby ending the proof. □

In the case of odd k, the existence of these solutions to constrained BKP systems was shown in [14] in a different fashion. In this case the conditions $C_{i,j+1} = (-)^k C_{i+1,j}$ for the integration constants reduce to the remarks made in [14] concerning these solutions. If k is even, the conditions reduce to $C_{i,j+1} = C_{i+1,j}$ and imply that all integration constants are zero: for example, $C_{1,2} = C_{2,1}$ but $C_{1,2} = -C_{2,1}$ (because of the anti-symmetry $\Omega(f, g) = -\Omega(g, f)$), which yields $C_{1,2} = 0$. Note that since $f_{i+1} = f_{i,kx}$ (with $k : 2, 4, \ldots$), all eigenfunction potentials in $(1, \ldots, N)$ are of the form $\Omega(f, f_{nx})$ where n is even and the absence of the integration constant now just means that $\Omega(f, f_{2x}) = f_x^2 - f f_{2x}$ and similarly for $\Omega(f, f_{4x})$, $\Omega(f_{2x}, f_{4x}) \ldots$.

6. Conclusions

A class of dimensional reductions of the BKP and CKP hierarchies is described. These are not classical symmetry reductions as there is no corresponding symmetry of the original BKP or CKP equations. They are conditional symmetry reductions in the sense that a new evolution is introduced which preserves the defining BKP/CKP condition only if it is immediately reduced. Among the reductions one finds the vector-modified KdV system and coupled KdV systems.

In the case of the BKP reductions, the scalar case ($n = 1$) of equation (20) was also derived (from the BKP hierarchy) in [19] via the method of density constraints (in which a linear combination of conserved densities is constrained to zero). One of the reasons for the author of [19] to study density constraints is that he was unable to find an expression for an eigenfunction symmetry for the BKP hierarchy (one was constructed in [14] containing two BKP eigenfunctions: $(q_x r - q r_x)_x$ solves the linearized BKP equation). The author is then forced to use a constraint ansatz $u = f(\Phi, \Phi_x, \ldots)$ (where f is to be determined from consistency conditions), which obscures the connection with the symmetry structure of the hierarchy.

In the method presented above, one can simply start form the KP (eigenfunction) symmetry structure and find classical or conditional symmetry reductions of BKP. In this manner the construction presented here actually unifies the description of classical symmetry constraints of BKP ($u_{t_k} = (q_x r - q r_x)_x$ and conditional symmetry conditions ($u = qr$ etc) in the single expression (35).

Let me further remark that the conditional symmetry reductions of CKP are defined in a completely analogous fashion. As there is a need for *two* eigenfunctions in the CKP constraint $L^k = B_k + q\partial^{-1}r - r\partial^{-1}q$, a simple ansatz of type $u = f(q, q_x, \ldots)$ (with only one eigenfunction) will not reveal the above constraints (i.e. will only yield the classical reductions in [15]).

It would be interesting to find a description of the present reductions in terms of an elementary constraint on the tau-functions.

I wish to remark that, as a given (1 + 1)-dimensional system is not necessarily a unique reduction of a (2 + 1)-dimensional system, it follows that there may exist different bilinear formulations and different representations of their solutions (Wronskian, Pfaffian etc) inherited from the various reduction procedures. This fact is demonstrated here in the case of the KdV equations, which arise more than once as a reduction of the BKP and CKP equations.

For the solutions of the reduced BKP hierarchy a new technique is introduced. This method allows for the direct verification of Pfaffian-type solutions on bilinear identities of the BKP and reduced BKP type. It has the advantage that all Hirota bilinear PDEs in the hierarchy can be verified in a single calculation. (When one checks the Hirota equations one at a time, the identities of Pfaffians used become more involved at higher order.) In this way a class of solutions is derived for the reduced systems. See also [7, 9] for a similar situation with the KP Wronskian and Grammian solutions.

Solutions to the coupled KdV system (12) are reported in [23].

Acknowledgments

The author is a post-doctoral fellow at the FWO-Vlaanderen (Belgium) and acknowledges the financial support extended within the framework of the interuniversity 'Poles of attraction' programme, contract no P4/08—Belgian State, Prime Minister's Office—Federal Office for Scientific, Technical and Cultural Affairs.

References

[1] Clarkson P and Kruskal M D 1989 New similarity reductions of the Boussinesq equation *J. Math. Phys.* **30** 2201–13

[2] Levi D and Winternitz P 1989 Non classical symmetry reductions: example of the Boussinesq equation *J. Phys. A: Math. Gen.* **22** 2915–24

[3] Orlov A Y 1991 Volterra operator algebra for zero curvature representation. Universality of KP *Nonlinear Processes in Physics* ed A S Fokas, D J Kaup, A C Newell and V E Zakharov (Berlin: Springer) pp 126–31

[4] Konopelchenko B and Strampp W 1991 The AKNS hierarchy as symmetry constraint of the KP hierarchy *Inverse Problems* **7** L17–24

[5] Cheng Y and Li Y S 1991 The constraint of the Kadomtsev–Petviashvili equation and its special solutions *Phys. Lett.* A **157** 22–6

[6] Cheng Y 1992 Constraints of the Kadomtsev–Petviashvili hierarchy *J. Math. Phys.* **33** 3774–82

[7] Loris I and Willox R 1997 On solutions of constrained KP equations *J. Math. Phys.* **38** 283–91

[8] Loris I and Willox R 1997 Bilinear form and solutions of the k-constrained Kadomtsev–Petviashvili hierarchy *Inverse Problems* **13** 411–20

[9] Loris I and Willox R 1997 KP symmetry reductions and a generalized constraint *J. Phys. A: Math. Gen.* **30** 6925–38

[10] Loris I and Willox R 1997 On solutions of constrained Kadomtsev–Petviashvili equations: Grammians *J. Math. Phys.* **38** 5190–7

[11] Aratyn H, Nissimov E and Pacheva S 1997 Constrained KP hierarchies: additional symmetries, Darboux–Bäcklund solutions and relations to multi-matrix models *Int. J. Mod. Phys.* A **12** 1265–340

[12] van de Leur J W 1997 The vector k-constrained KP hierarchy and Sato's Grassmannian *J. Geom. Phys.* **23** 83–96

[13] Willox R and Loris I 1999 KP constraints from reduced multi-component hierarchies *J. Math. Phys.* **40** 6501–25

[14] Loris I and Willox R 1999 Symmetry reductions of the BKP hierarchy *J. Math. Phys.* **40** 1420–31

[15] Loris I 1999 On reduced CKP equations *Inverse Problems* **15** 1099–109

[16] Iwao M and Hirota R 1997 Soliton solutions of a coupled modified KdV equations *J. Phys. Soc. Japan* **66** 577–88

[17] Kadomtsev B B and Petviashvili V I 1970 On the stability of solitary waves in weakly dispersing media *Sov. Phys.–Dokl.* **15** 539–41

[18] Date E, Kashiwara M, Jimbo M and Miwa T 1983 Transformation groups for soliton equations *Proc. RIMS Symp. on Non-Linear Integrable Systems—Classical Theory and Quantum Theory* ed M Jimbo and T Miwa (Singapore: World Scientific) pp 39–119

[19] Zeng Y 1991 The constraint on potential and decomposition for $(2+1)$-dimensional integrable systems *J. Phys. A: Math. Gen.* **24** L1065–72

[20] Gardner C S, Greene J M, Kruskal M D and Miura R M 1967 Method for solving the Korteweg–de Vries equation *Phys. Rev. Lett.* **19** 1095–7

[21] Date E, Jimbo M, Kashiwara M and Miwa T 1981 KP hierarchies of orthogonal and symplectic type—transformation groups for soliton equations VI *J. Phys. Soc. Japan* **50** 3813–8

[22] Jimbo M and Miwa T 1983 Solitons and infinite-dimensional Lie algebras *Publ. RIMS Kyoto Univ.* **19** 943–1001

[23] Loris I 2001 Solutions of coupled Korteweg–de Vries systems *J. Phys. Soc. Japan* **70** 622–5

[24] Sawada K and Kotera T 1974 A method for finding N-soliton solutions of the KdV equation and KdV-like equations *Prog. Theor. Phys.* **51** 1355–67

[25] Hirota R and Satsuma J 1981 Soliton solutions of a coupled Korteweg–de Vries equation *Phys. Lett.* A **85** 407–8

[26] Tam H W, Ma W X, Hu X B and Wang D L 2000 The Hirota–Satsuma coupled KdV equation and a coupled Ito system revisited *J. Phys. Soc. Japan* **69** 45–52

[27] Nimmo J J C and Willox R 1997 Darboux transformations for the two-dimensional Toda lattice *Proc. R. Soc.* A **453** 2497–525

[28] Hirota R 1980 Direct methods in soliton theory *Solitons* ed R K Bullough and P J Caudrey (Berlin: Springer) pp 157–76

[29] Dickey L 1991 *Soliton Equations and Hamiltonian Systems (Avanced Series in Mathematical Physics vol 12)* (Singapore: World Scientific)

[30] Hirota R 1989 Soliton solutions to the BKP equations I. The Pfaffian technique *J. Phys. Soc. Japan* **58** 2285–96

[31] Nimmo J J C 1990 Hall–Littlewood symmetric functions and the BKP hierarchy *J. Phys. A: Math. Gen.* **23** 751–60